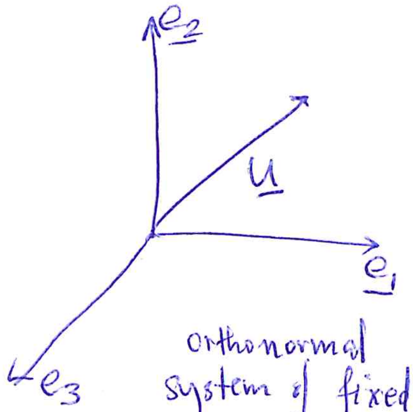
CMM Exercise 01 — 10.04.2024

Einstein Summation Convention.



orthonormal system of fixed basis vectors

$$\underline{u} = u_1 \underline{e_1} + u_2 \underline{e_2} + u_3 \underline{e_3}$$
$$= \sum_{i=1}^{n} u_i \underline{e_i}$$

$i, j, k = 1 \dots n$
where $\mathbb{R}^n$

$$\underline{u} = u_i e_i$$

Repeated index stands for summation

__Ex 1.__ $$\frac{\partial f_i}{\partial x_j} dx_j = \frac{\partial f_i}{\partial x_1} dx_1 + \frac{\partial f_i}{\partial x_2} dx_2 + \frac{\partial f_i}{\partial x_3} dx_3$$

__Ex 2__ $$A_{ij} B_{jk} = A_{i1} B_{1k} + A_{i2} B_{2k} + A_{i3} B_{3k}$$

* Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

eg. $\delta_{11} = 1$

$\delta_{13} = 0$

* Levi-Cevita Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even index permutations } [(123), (312), (231)] \\ -1 & \text{for odd index permutations } [(321), (132), (213)] \\ 0 & \text{otherwise} \end{cases}$$

__Ex 3__

$$u_i v_j \delta_{ij} = \underline{u_1 v_j \delta_{1j}} + \underline{u_2 v_j \delta_{2j}} + \underline{u_3 v_j \delta_{3j}}$$
$$= u_1 v_1 \cancel{\delta_{11}}^{1} + \cancel{u_2 v_1 \delta_{21}}^{0} + \cancel{u_3 v_1 \delta_{31}}^{0}$$
$$+ \cancel{u_2 v_1 \delta_{21}}^{0} + u_2 v_2 \cancel{\delta_{22}}^{1} + \cancel{u_2 v_3 \delta_{23}}^{0}$$
$$+ \cancel{u_3 v_1 \delta_{31}}^{0} + \cancel{u_3 v_2 \delta_{32}}^{0} + u_3 v_3 \cancel{\delta_{33}}^{1}$$
$$= u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i$$

$$\epsilon_{ijk} u_i v_j \underline{e_k} = (u_2 v_3 - v_3 u_2) \underline{e_1} + (u_3 v_1 - u_1 v_3) \underline{e_2} + (u_1 v_2 - u_2 v_1) \underline{e_3}$$

* Index Notation for Tensors

$$\underline{\underline{A}} = A_{ij}(\underline{e}_i \otimes \underline{e}_j)$$
$$= A_{11}(\underline{e}_1 \otimes \underline{e}_1) + A_{12}(\underline{e}_1 \otimes \underline{e}_2) + A_{13}(\underline{e}_1 \otimes \underline{e}_3)$$
$$+ A_{21}(\underline{e}_2 \otimes \underline{e}_1) + A_{22}(\underline{e}_2 \otimes \underline{e}_2) + A_{23}(\underline{e}_2 \otimes \underline{e}_3)$$
$$+ A_{31}(\underline{e}_3 \otimes \underline{e}_1) + A_{32}(\underline{e}_3 \otimes \underline{e}_2) + A_{33}(\underline{e}_3 \otimes \underline{e}_3)$$

$$(\underline{a} \otimes \underline{b})\,\underline{c} = \underline{d}$$
$$(\underline{b} \cdot \underline{c})\,\underline{a} = \underline{d}$$
$$(b_i c_i)\,a_j = d_j$$

Index & Component Notation for Differential Operators

$\underline{\nabla} \rightarrow$ Nabla : Differential Vector Operator

$\underline{\nabla}(\bullet)$

Component Notation

$$\underline{\nabla} = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$$

$$\underline{\nabla} u = \underline{i}\frac{\partial u}{\partial x} + \underline{j}\frac{\partial u}{\partial y} + \underline{k}\frac{\partial u}{\partial z}$$

Ex 4

Try Yourself:

$\underline{\nabla} \cdot \underline{u} \rightarrow$ Divergence $\rightarrow$ div $\underline{u}$

$\underline{\nabla} \times \underline{u} \rightarrow$ Curl $\underline{u}$

Index Notation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \text{ or } \begin{pmatrix} r \\ \theta \\ z \end{pmatrix}$$

Cartesian CS   Spherical CS   Cylindrical CS

Gradient of Scalar Field

$$\underline{\nabla} u = \partial_i u\, \underline{e}_i = \frac{\partial u}{\partial x_i} = \underline{e}_1\frac{\partial u}{\partial x_1} + \underline{e}_2\frac{\partial u}{\partial x_2} + \underline{e}_3\frac{\partial u}{\partial x_3}$$

* Gradient of Vector Field.

$$\nabla \underline{v} = \partial_j v_i = \frac{\partial v_j}{\partial x_i} \underline{e_i \otimes e_j}$$

* Divergence of Vector Field

$$\underline{\nabla} \cdot \underline{v} = \partial_i v_i = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$$
$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

* Curl of Vector Field

$$\underline{\nabla} \times \underline{v} = \epsilon_{ijk}\, \partial_i v_j\, \underline{e_k}$$

* Note : Ordering of Terms is important when ∇ is used

$$\underline{\nabla} \cdot \underline{a} \longrightarrow \partial_i a_i \longrightarrow \text{scalar}$$

$$\underline{a} \cdot \underline{\nabla}(\circ) \longrightarrow a_i \partial_i(\circ) \longrightarrow \text{vector operator}$$

Ex 5: Momentum Balance

$$\boxed{\rho \frac{\partial v}{\partial t} + \rho\, \underline{v} \cdot \underline{\nabla}\, \underline{v} = \underline{\nabla} \cdot \underline{\underline{\sigma}} + \rho \underline{b}}$$

$$\Rightarrow \rho \frac{D\underline{v}}{Dt} = \underline{\nabla}\, \underline{\underline{\sigma}} + \rho \underline{b}$$

$\frac{D(\circ)}{Dt} \rightarrow$ Material Time Derivative

$$\frac{D(\circ)}{Dt} = \frac{\partial}{\partial t}(\circ) + \underline{v} \cdot \underline{\nabla}(\circ)$$

Ex 6.
Try Yourself :
$\frac{D(\circ)}{Dt}$ in a) Index Notation
b) Component Form

* Component Form

$$x: \rho \frac{\partial v_x}{\partial t} + \rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho b_x$$

$$y: \rho \frac{\partial v_y}{\partial t} + \rho\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho b_y$$

$$z: \rho \frac{\partial v_z}{\partial t} + \rho\left(v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho b_z$$

* Index Notation : $\rho\, \partial_t v_i + \rho\, v_j \partial_j v_i = \partial_j \sigma_{ji} + \rho b_i$